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Analytic and algebraic properties of Riccati equations: A survey

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Abstract

This is a survey of recent results on the classical problems of the *analytic properties of Riccati equations* and *algebraic properties of Riccati equations* and applications to spatially distributed systems.

Keywords: Linear systems; Infinite-dimensional systems; Spatially invariant systems; Platoons; Spatially distributed systems

1. Motivation: spatially distributed systems

In this article we consider two classical problems for solutions to Riccati equations:

• *Analytic property:* Suppose that $A_1(z), A_2(z), D(z), M(z)$ are analytic $n \times n$ matrix functions of the complex variable z on a connected open subset Ω of the complex plane $\mathbb{C}$. Under what conditions will the Riccati equation

$$A_1(z)Q(z) + Q(z)A_2(z) - Q(z)D(z)Q(z) + M(z) = 0,$$

have a solution $Q(z)$ that is analytic in a subset of Ω?

• *Algebraic property:* Let R be a Banach algebra with the involution $\cdot^\star$. Given $A \in R^{n\times n}$, $B \in R^{m\times n}$, $A \in R^{n\times p}$, under what conditions will the algebraic Riccati equation

$$A^\star Q + QA - QBB^\star Q + C^\star C = 0 \tag{1}$$

have a solution $Q = Q^\star$ in R?

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These problems have been considered earlier from a mathematical perspective. However, as has often been the case in systems theory, new engineering applications give rise to new mathematical formulations and new insights into the theory. We illustrate this phenomenon with the following class of systems:

$$\dot{z}_r(t) = \sum_{l\in\mathbb{Z}} A_{rl} z_l(t) + \sum_{l\in\mathbb{Z}} B_{rl} u_l(t), \tag{2}$$

$$y_r(t) = \sum_{l\in\mathbb{Z}} C_{rl} z_l(t) + \sum_{l\in\mathbb{Z}} D_{rl} u_l(t), \tag{3}$$

where $r, l \in \mathbb{Z}$, the set of integers, $A_{rl} \in \mathbb{C}^{n\times n}$, $B_{rl} \in \mathbb{C}^{n\times m}$, $C_{rl} \in \mathbb{C}^{p\times n}$, $D_{rl} \in \mathbb{C}^{p\times m}$ and $z_r(t) \in \mathbb{C}^n$, $u_r(t) \in \mathbb{C}^m$ and $y_r(t) \in \mathbb{C}^p$ are the state, the input and the output vectors, respectively, at time $t \geq 0$ and spatial point $r \in \mathbb{Z}$. This class belongs to the class of *spatially distributed systems* introduced in Bamieh et al. [1]. In fact, they form a special class of infinite-dimensional systems.

Using the terminology and formalism of Curtain and Zwart [12], Eqs. (2) and (3) can be formulated as an infinite-dimensional linear system $\Sigma(A, B, C, D)$

$$\begin{aligned} \dot{z}(t) &= (Az)(t) + (Bu)(t), \\ y(t) &= (Cz)(t) + (Du)(t), \quad t \geq 0, \end{aligned} \tag{4}$$

with the state space $Z = \ell_2(\mathbb{C}^n)$, the input space $U = \ell_2(\mathbb{C}^m)$ and the output space $Y = \ell_2(\mathbb{C}^p)$.

In the special case where $A_{rl} = A_{r-l}$ the system (2), (3) reduces to

$$\dot{z}_r(t) = \sum_{l\in\mathbb{Z}} A_l z_{r-l}(t) + \sum_{l\in\mathbb{Z}} B_l u_{r-l}(t), \tag{5}$$

$$y_r(t) = \sum_{l\in\mathbb{Z}} C_l z_{r-l}(t) + \sum_{l\in\mathbb{Z}} D_l u_{r-l}(t). \tag{6}$$

The corresponding operators A, B, C, D in the formulation (4) are convolution operators. Denoting the signals and the convolution operators generically by $x(t)$ and T, respectively, we have

$$((Tx)(t))_r = \sum_{l\in\mathbb{Z}} T_l x_{r-l}(t) = \sum_{l\in\mathbb{Z}} T_{r-l} x_l(t).$$

The T are shift-invariant operators in $\ell_2(\mathbb{C}^n)$ which in the engineering literature are called the *spatially invariant* operators. One of the motivations for studying these spatially invariant systems stems from the interest shown in the engineering literature in controlling infinite platoons of vehicles over the years [17–19,4,16]. We shall call these systems *platoon-type systems*.

Taking (formally) discrete Fourier transforms $\mathfrak{F} : \ell_2(\mathbb{C}^n) \to \mathbf{L}_2(\mathbb{T}; \mathbb{C}^n)$ of the system equations (4), where $\mathbb{T}$ denotes the unit circle, we obtain

$$\begin{aligned} \dot{\check{z}}(t) &= \mathfrak{F}\dot{z}(t) = \check{A}\check{z}(t) + \check{B}\check{u}(t), \\ \check{y}(t) &= \mathfrak{F}y(t) = \check{C}\check{z}(t) + \check{D}\check{u}(t). \end{aligned} \tag{7}$$

Note that $\check{z}$ denotes the Fourier transform $\mathfrak{F}z$ of z and $\check{A} = \mathfrak{F}A\mathfrak{F}^{-1}$, $\check{B} = \mathfrak{F}B\mathfrak{F}^{-1}$, $\check{C} = \mathfrak{F}C\mathfrak{F}^{-1}$ and $\check{D} = \mathfrak{F}D\mathfrak{F}^{-1}$ are shift-invariant operators. If $\check{A}, \check{B}, \check{C}, \check{D} \in \mathbf{L}_\infty(\mathbb{T}; \mathbb{C}^{\bullet\times\bullet})$, then A, B, C, D are all bounded operators ("$\bullet$" denotes the appropriate dimension). We shall assume this throughout this article. Consequently, $\check{A}$ is a multiplicative operator of the form

$$\check{A}(\phi) := \sum_{l\in\mathbb{Z}} A_l \phi^{-l} \quad \text{for almost all } \phi \in \mathbb{T}. \tag{8}$$

Note that the linear system $\Sigma(A, B, C, D)$ on the state space $\ell_2(\mathbb{C}^n)$ with input and output spaces $\ell_2(\mathbb{C}^m)$ and $\ell_2(\mathbb{C}^m)$, respectively, is isometrically isomorphic to the linear system $\Sigma(\check{A}, \check{B}, \check{C}, \check{D}) = \Sigma(\mathfrak{F}A\mathfrak{F}^{-1}, \mathfrak{F}B\mathfrak{F}^{-1}, \mathfrak{F}C\mathfrak{F}^{-1}, \mathfrak{F}D\mathfrak{F}^{-1})$ on the state space $\mathbf{L}_2(\mathbb{T}; \mathbb{C}^n)$ with input and output spaces $\mathbf{L}_2(\mathbb{T}; \mathbb{C}^m)$ and $\mathbf{L}_2(\mathbb{T}; \mathbb{C}^p)$, respectively. Their system theoretic properties are identical (see [12, Exercise 2.5]) and so it suffices to apply the standard theory from [12] to this class of infinite-dimensional systems.

The Fourier transformed systems (7) have a pointwise interpretation for almost all $\phi \in \mathbb{T}$:

$$\begin{aligned} \frac{\partial}{\partial t}\check{z}(\phi, t) &= \check{A}(\phi)\check{z}(\phi, t) + \check{B}(\phi)\check{u}(\phi, t) \\ \check{y}(\phi, t) &= \check{C}(\phi)\check{z}(\phi, t) + \check{D}(z)\check{u}(\phi, t), \quad t \geq 0. \end{aligned} \tag{9}$$

When $\check{A}, \check{B}, \check{C}, \check{D}$ are continuous, this class of infinite-dimensional systems is equivalent to infinitely many finite-dimensional systems parameterized by $\phi \in \mathbb{T}$. It turns out that the analysis of system theoretic properties of $\Sigma(\check{A}, \check{B}, \check{C}, \check{D})$ can be deduced from those of the finite-dimensional systems $\Sigma(\check{A}(\phi), \check{B}(\phi), \check{C}(\phi), \check{D}(\phi))$ for all $\phi \in \mathbb{T}$ (see Section 2). That leads to the following control design based on the well-known linear quadratic regulator theory.

Theorem 1.1. *Suppose that $\check{A}(\phi), \check{B}(\phi), \check{C}(\phi)$ are continuous in ϕ on $\mathbb{T}$.*
If $(\check{A}(\phi), \check{B}(\phi))$ is stabilizable and $(\check{A}(\phi), \check{C}(\phi))$ is detectable for each $\phi \in \mathbb{T}$, then for each $\phi \in \mathbb{T}$ the following family of Riccati equations has a unique symmetric solution $\check{Q}(\phi)$:

$$\check{A}(\phi)^*\check{Q}(\phi) + \check{Q}(\phi)\check{A}(\phi) - \check{Q}(\phi)\check{B}(\phi)\check{B}(\phi)^*\check{Q}(\phi) + \check{C}(\phi)^*\check{C}(\phi) = 0. \tag{10}$$

Moreover, $\check{A}(\phi) - \check{B}(\phi)\check{B}(\phi)^*\check{Q}(\phi)$ is stable and $\check{Q}(\phi)$ is positive semidefinite for each $\phi \in \mathbb{T}$.

The stabilizing feedback in Theorem 1.1 is $\check{K} = -\check{B}^*\check{Q}$ with the form

$$\check{K}(\phi) = \sum_{l\in\mathbb{Z}} K_l \phi^{-l}, \quad \phi \in \mathbb{T}$$

and the control action has the form

$$u(t)_r = \sum_{l\in\mathbb{Z}} K_{r-l} z_l(t) = \sum_{l\in\mathbb{Z}} K_l z_{r-l}(t).$$

Even when $\check{A}(\phi), \check{B}(\phi), \check{C}(\phi)$ have finitely many nonzero Fourier coefficients in the expansion (8), $\check{K}$ will typically have infinitely many nonzero Fourier coefficients. For practical implementation it is desirable that the control action only depend on the nearest neighbours $z_r, z_{r\pm1}, \ldots, z_{r\pm(r+s)}$, where s can be chosen as small as possible. One possibility is to approximate $\check{K}$ by its truncation. However, this will only work if the Fourier coefficients k_r of $\check{K}$ decay rapidly as $r \to \infty$. In general, one only knows that

$$\sup_{\phi\in\mathbb{T}} \|\check{K}(\phi)\|_{\mathbb{C}^{n\times n}} < \infty$$

(see Section 2). Consequently, it is essential to find extra conditions under which the solutions of Riccati equations will have a *spatially decaying property*. This requirement has motivated the study of analytic properties of solutions to the Riccati equation (see Section 3) and also the algebraic properties (see Sections 4 and 5).

2. Properties of platoon-type systems

Recall that we are assuming that $\check{A}$ is bounded on $\boldsymbol{L}_2(\mathbb{T}; \mathbb{C}^{n\times n})$, and so $\check{A} \in \boldsymbol{L}_\infty(\mathbb{T}; \mathbb{C}^{n\times n})$ and $\|\check{A}\|_\infty = \text{ess sup}_{\phi\in\mathbb{T}} \|\check{A}(\phi)\|_{\mathbb{C}^{n\times n}}$. In this section we summarize the main system theoretic properties of $\Sigma(\check{A}, \check{B}, \check{C}, -)$. As predicted in Bamieh et al. [1], they can be verified by standard tests on the matrices $\check{A}(\phi), \check{B}(\phi), \check{C}(\phi)$ for each parameter $\phi \in \mathbb{T}$. Proofs can be found in Curtain et al. [9].

Since $\check{A}$ is bounded, the semigroup $e^{\check{A}t}$ is *exponentially stable*, i.e.,

$$\|e^{\check{A}t}\|_\infty \le Me^{-\alpha t} \quad \text{for } \alpha, M > 0$$

if and only if

$$\sup\{\text{Re}(\lambda) \mid \lambda \in \sigma(\check{A})\} < 0.$$

In particular, if $\check{A}(\phi)$ is continuous for $\phi \in \mathbb{T}$, then a necessary and sufficient condition for exponential stability of $e^{\check{A}t}$ is

$$\sup\{\text{Re}(\lambda) \mid \exists\phi \in \mathbb{T} \text{ s.t. } \det(\lambda I - \check{A}(\phi)) = 0\} < 0.$$

The weaker property of *strong stability*

$$e^{\check{A}t}\check{z} \to 0 \quad \text{for all } \check{z} \in \boldsymbol{L}_2(\mathbb{T}; \mathbb{C}^{n\times n})$$

is not determined by the spectrum of $\check{A}$ alone. However, when the strongly continuous semigroup $e^{\check{A}t}$ is uniformly bounded in norm for $t \ge 0$, i.e.,

$$\sup_{t\ge 0} \operatorname*{ess\,sup}_{\phi\in\mathbb{T}} \|e^{\check{A}(\phi)t}\|_{\mathbb{C}^{n\times n}} < \infty, \tag{11}$$

then $e^{\check{A}t}$ is strongly stable if and only if for almost all $\phi \in \mathbb{T}$ the finite-dimensional semigroups $\{e^{\check{A}(\phi)t} \mid \phi \in \mathbb{T}\}$ are exponentially stable.

We recall the definitions of controllability and observability from [12, Definitions 4.1.3, 4.1.12].

Definition 2.1. 1. $\Sigma(\check{A}, \check{B}, -, -)$ is *exactly controllable* on $[0, \tau]$ (for $\tau > 0$) if all points in Z can be reached from the origin at time τ;
2. $\Sigma(\check{A}, \check{B}, -, -)$ is *approximately controllable* on $[0, \tau]$ (for $\tau > 0$) if given an arbitrary $\varepsilon > 0$ it is possible to steer from the origin to within a distance ε from all points in the state space at time τ;
3. $\Sigma(\check{A}, -, \check{C}, -)$ exactly observable on $[0, \tau]$ (for $\tau > 0$) if the initial state can be uniquely and continuously constructed from the knowledge of the output in $\boldsymbol{L}_2([0, \tau]; Y)$;
4. $\Sigma(\check{A}, -, \check{C}, -)$ is *approximately observable* on $[0, \tau]$ (for $\tau > 0$) if knowledge of the output in $\boldsymbol{L}_2([0, \tau]; Y)$ determines the initial state uniquely.

These properties can be checked using finite-dimensional tests for $\phi \in \mathbb{T}$.

Lemma 2.2. *$\Sigma(\check{A}, \check{B}, -, -)$ is approximately controllable on $[0, \tau]$ for all $\tau > 0$ if and only if $(\check{A}(\phi), \check{B}(\phi))$ is controllable for almost all $\phi \in \mathbb{T}$.*

$\Sigma(\check{A}, -, \check{C}, -)$ is approximately observable on $[0, \tau]$ for all $\tau > 0$ if and only if $(\check{A}(\phi), \check{C}(\phi))$ is observable for almost all $\phi \in \mathbb{T}$.

If $\check{A}(\phi)$ and $\check{B}(\phi)$ are continuous in ϕ on $\mathbb{T}$, then $\Sigma(\check{A}, \check{B}, -, -)$ is exactly controllable on $[0, \tau]$ if and only if $(\check{A}(\phi), \check{B}(\phi))$ is controllable for all $\phi \in \mathbb{T}$.

If $\check{A}(\phi)$ and $\check{C}(\phi)$ are continuous in ϕ on $\mathbb{T}$, then $\Sigma(\check{A}, -, \check{C}, -)$ is exactly observable on $[0, \tau]$ if and only if $(\check{A}(\phi), \check{C}(\phi))$ is observable for all $\phi \in \mathbb{T}$.

We recall that $\Sigma(\check{A}, \check{B}, \check{C}, -)$ is exponentially stabilizable if there exists a $\check{F} \in \mathcal{L}(Z, U)$ such that $\check{A} + \check{B}\check{F}$ is exponentially stable and it is exponential detectable if there exists a $\check{L} \in \mathcal{L}(Y, Z)$ such that $\check{A} + \check{L}\check{C}$ is exponentially stable.

Lemma 2.3. *Suppose that $\check{A}(\phi), \check{B}(\phi), \check{C}(\phi)$ are continuous in ϕ on $\mathbb{T}$.*

$\Sigma(\check{A}, \check{B}, \check{C}, 0)$ is exponentially stabilizable if and only if $(\check{A}(\phi), \check{B}(\phi))$ is exponentially stabilizable for each $\phi \in \mathbb{T}$.

$\Sigma(\check{A}, \check{B}, \check{C}, 0)$ is exponentially detectable if and only if $(\check{A}(\phi), \check{C}(\phi))$ is exponentially detectable for each $\phi \in \mathbb{T}$.

As a consequence of these results, the standard application of the infinite-dimensional Riccati theory [12, Theorem 6.2.7] shows that the pointwise solution of (10) corresponds to the solution to the infinite-dimensional Riccati equation

$$\check{A}^*\check{Q} + \check{Q}\check{A} - \check{Q}\check{B}\check{B}^*\check{Q} + \check{C}^*\check{C} = 0, \tag{12}$$

and $\check{Q} \in \boldsymbol{L}_\infty(\mathbb{T}; \mathbb{C}^{n\times n})$. Under the same assumptions the dual filter Riccati equation also has a unique nonnegative solution $\check{P} \in \boldsymbol{L}_\infty(\mathbb{T}; \mathbb{C}^{n\times n})$;

$$\check{A}\check{P} + \check{P}\check{A}^* - \check{P}\check{C}^*\check{C}\check{P} + \check{B}\check{B}^* = 0. \tag{13}$$

Clearly, similar remarks hold for other types of Riccati equations, including H_∞-type Riccati equations.

3. Analytic properties of Riccati equation solutions

In Section 1 we saw that it is important to find extra conditions under which the feedback $\check{K} = -\check{B}^*\check{Q}$, where $\check{Q}$ is the solution to the Riccati equation (12), has a spatially decaying property.

A sufficient condition is that $\check{K}$ have an analytic extension to the annulus around the unit circle:

$$\mathbb{A}(\tau) = \{z \in \mathbb{C} \mid e^{-\tau} < |z| < e^{\tau}\}, \quad \tau > 0. \tag{14}$$

For then when $K(z) = \sum_{l\in\mathbb{Z}} K_l z^{-l}$ is the Laurent series for $K(z)$, for every β, $0 \le \beta < \tau$, there exists a positive μ such that $\|K_l\| \le \mu e^{-|l|\beta}$.

So the Fourier coefficients of $\check{K}$ will decay exponentially fast if $\check{B}$ and the solution to the Riccati equation (10) have an analytic extension to an annulus around the unit circle. This idea was first exploited in an analogous situation in [1] to obtain conditions for analyticity of solutions to a modified Riccati equation. This motivated a deeper analysis of the analyticity of solutions to Riccati equations in Curtain and Rodman [10]. We summarize the main results applied to the particular case of platoon-type systems.

The analogue of the approach used in [1] is to seek an analytic extension $Q(z)$ of the solution $\check{Q}$ to the Riccati equation (12) to an annulus $\mathbb{A}(\tau)$, i.e., $Q(\phi) = \check{Q}(\phi)$ for $\phi \in \mathbb{T}$. The obvious candidate is a solution $Q(z)$ for $z \in \mathbb{A}(\tau)$ to the following nonstandard Riccati equation:

$$A^{\sim}(z)Q(z) + Q(z)A(z) - Q(z)B(z)B^{\sim}(z)Q(z) + C^{\sim}(z)C(z) = 0, \tag{15}$$

where $A^{\sim}(z) := A(\overline{z^{-1}})^*$ and we suppose that $A(z), B(z), C(z)$ are $n \times n, n \times m$ and $p \times n$ matrix valued functions. In fact, we can treat the more general equation

$$A^{\sim}(z)Q(z) + Q(z)A(z) - Q(z)D(z)Q(z) + M(z) = 0, \tag{16}$$

where $D(z) = D^{\sim}(z)$ and $Q(z) = Q^{\sim}(z)$ are $n \times n$. The corresponding Hamiltonian matrix function is given by

$$H(z) = \begin{bmatrix} A(z) & -D(z) \\ -M(z) & -A^{\sim}(z) \end{bmatrix}. \tag{17}$$

We define a *stabilizing solution* to (16) as a solution $Q(z)$, defined for all $z \in \mathbb{A}(\tau)$ for some $\tau > 0$, such that $Q^{\sim}(z) = Q(z)$ and $A(z) - D(z)Q(z)$ is stable for all $z \in \mathbb{A}(\tau)$.

Theorem 3.1. *Suppose that for some $\tau > 0$ the Hamiltonian matrix $H(z)$ is analytic in the annulus $\mathbb{A}(\tau)$, and the following conditions hold:*

(1) *For every $\phi \in \mathbb{T}$, the matrix $H(\phi)$ has no eigenvalues on the imaginary axis.*
(2) *For every $\phi \in \mathbb{T}$, there exists a stabilizing matrix solution.*

Then for some β, $0 < \beta \le \tau$ and for every $z \in \mathbb{A}(\beta)$ there exists a unique stabilizing solution $Q(z)$ of (16), and $Q(z)$ is analytic in $\mathbb{A}(\beta)$.

Theorem 3.1 holds for very general Riccati equations, including those of H_∞-type. For the special regulator Riccati equation (12) the following corollary applies.

Corollary 3.2. *Suppose that for some $\tau > 0$ the Hamiltonian matrix $H(z)$ is analytic in the annulus $\mathbb{A}(\tau)$, $(A(\phi), B(\phi))$ is stabilizable and $(A(\phi), C(\phi))$ is detectable for all $\phi \in \mathbb{T}$. Then there exists β, $0 < \beta \le \tau$, such that for every $z \in \mathbb{A}(\beta)$, there exists a unique stabilizing solution $Q(z)$ of (15), and $Q(z)$ is analytic in $\mathbb{A}(\beta)$.*

The drawback of the above result is that it gives no information about the size of β. To obtain more information about the size of the annulus of analyticity we need to examine the existence of a stabilizing solution. The following result is an analogue of a previous result in [1].

Theorem 3.3. *Suppose that:*

1. *$(A(\phi), B(\phi), C(\phi))$ is stabilizable and detectable for every $\phi \in \mathbb{T}$;*
2. *for some $\tau > 0$ the matrix function $H(z)$ is analytic in the annulus $\mathbb{A}(\tau)$;*
3. *the matrix $H(z)$ has no eigenvalues on the imaginary axis for all $z \in \mathbb{A}(\tau)$;*
4. *$(A(z), B(z)B^{\sim}(z))$ is stabilizable for all $z \in \mathbb{A}(\tau)$;*
5. *for every $z \in \mathbb{A}(\tau)$, if for some vectors x, y there holds $y^T B(z)B^{\sim}(z)x = 0$, then $y^T B(z) = 0$ or $B^{\sim}(z)x = 0$.*

Then for all $z \in \mathbb{A}(\tau)$, (15) has a unique stabilizing solution $Q(z)$ that is analytic on $\mathbb{A}(\tau)$.

Note that condition 5 is very restrictive. The proof in [10] relies on condition 5 in a crucial manner and cannot be extended to more general Riccati equations. An open question is whether it is possible to prove a general result without condition 5 or, alternatively, construct a counterexample.

4. Riccati equations on commutative Banach algebras

Another approach to the spatially decaying question is to pose it as an algebraic one. A manner of measuring the decay rate of the Fourier coefficients of $\check{Q} \in \boldsymbol{L}_\infty(\mathbb{T}; \mathbb{C}^{n\times n})$ is to use the even-weighted Wiener algebras.

Example 4.1 (*Even-Weighted Wiener Algebras*)**.** Let $(\alpha_k)_{k\in\mathbb{Z}}$ be any sequence of even weights, that is, the α_k's are positive real numbers satisfying the following:

$$\alpha_{-k} = \alpha_k; \qquad \alpha_{k+l} \le \alpha_k \alpha_l \quad k, l \in \mathbb{Z}.$$

Consider the even-weighted Wiener algebra $W_\alpha(\mathbb{T})$ of the unit circle $\mathbb{T}$ given by

$$W_\alpha(\mathbb{T}) = \left\{ f : f(\phi) = \sum_{k\in\mathbb{Z}} f_k \phi^k, \phi \in \mathbb{T} \text{ and } \sum_{k\in\mathbb{Z}} \alpha_k |f_k| < +\infty \right\},$$

with pointwise operations, and the norm

$$\|f\|_{W_\alpha(\mathbb{T})} = \sum_{k\in\mathbb{Z}} \alpha_k |f_k|, \qquad f(\phi) = \sum_{k\in\mathbb{Z}} f_k \phi^k, \quad z \in \mathbb{T}.$$

Then this is a Banach algebra; see [14, Section 19.4, pp. 118–120]. The maximal ideal space of these even-weighted Wiener algebras can be identified with the closed annulus

$$\overline{\mathbb{A}(\rho)} = \{z \in \mathbb{C} : 1/\rho \le |z| \le \rho\},$$

where $\rho := \inf_{k>0} \sqrt[k]{\alpha_k} = \lim_{k\to\infty} \sqrt[k]{\alpha_k}$. The Gelfand transform is given by

$$\widehat{f}(z) = \sum_{k\in\mathbb{Z}} f_k z^k, \quad z \in \overline{\mathbb{A}(\rho)}.$$

So $\hat{f}$ is an analytic extension of f to the open annulus $\mathbb{A}(\rho)$.

Examples that occur in applications are exponential weights

$$\alpha_k = e^{\alpha|k|}, \quad \alpha > 0,$$

subexponential weights

$$\alpha_k = e^{\alpha|k|^\beta}, \quad \alpha > 0, 0 \le \beta < 1,$$

and polynomial weights

$$\alpha_k = (1 + |k|)^s, \quad s \ge 0.$$

When $\rho = 1$ the weights are said to satisfy the *Gelfand–Raikov–Shilov condition* and the annulus $\mathbb{A}(\rho)$ degenerates to the circle $\mathbb{T}$. Subexponential weights and polynomial weights satisfy this condition, but exponential weights do not.

The choice of involution is not unique. One choice is

$$\hat{f}^{\sim}(z) = \overline{\hat{f}\left(\frac{1}{\overline{z}}\right)}, \quad z \in \overline{\mathbb{A}(\rho)}.$$

Another choice would be

$$\hat{f}^{\dagger}(z) := \overline{\hat{f}(\overline{z})}, \quad z \in \overline{\mathbb{A}(\rho)}.$$

Note that under the Gelfand–Raikov–Shilov condition the involution $\cdot^{\sim}$ reduces to the following:

$$f^{\sim}(\phi) = f(1/\phi), \quad \phi \in \mathbb{T}, f \in W_\alpha(\mathbb{T}).$$

For simplicity of notation we denote the Gelfand transform of the matrix function $\check{A} \in W_\alpha(\mathbb{T})$ by $\hat{A}$ which is defined on $\overline{\mathbb{A}(\rho)}$.

The obvious question is whether $\check{Q} \in W_\alpha(\mathbb{T})$ will hold under the assumptions that the components $\check{A}, \check{B}, \check{C} \in W_\alpha(\mathbb{T})$ and the Riccati equation (12) has a solution in $\boldsymbol{L}_\infty(\mathbb{T}; \mathbb{C}^{n\times n})$. The following example from [5] shows that this is not always the case.

Example 4.2. Consider the Riccati equation (12) with $\check{A} = 0$, $\check{B}(\phi) = 10 - \phi - 1/\phi$, $\check{C} = 1$, which are in $W_\alpha(\mathbb{T})$ for arbitrary even weights $(\alpha_k)_{k\in\mathbb{Z}}$.

The Riccati equation has the unique positive solution

$$\check{Q}(\phi) = \frac{1}{10 - \phi - 1/\phi} = \frac{1}{4\sqrt{6}} \sum_{k\in\mathbb{Z}} \delta^{-|k|}\phi^k, \quad \delta = 5 + \sqrt{24}.$$

It is in $W_\alpha(\mathbb{T})$ for exponential weights $e^{\alpha|k|}$ provided $e^\alpha < \delta$, but it is clearly not in $W_\alpha(\mathbb{T})$ for arbitrary weights.

Despite this disappointing example, it is possible to give conditions for Riccati equations to have nice algebraic properties with respect to commutative *-Banach algebras. We summarize the main results from Curtain and Sasane [11].

R will denote a commutative, unital, complex, semisimple Banach algebra, which possesses an involution $\cdot^\star$.

Denote the usual adjoint of a matrix $M = [m_{ij}] \in \mathbb{C}^{p\times m}$ by $M^* \in \mathbb{C}^{m\times p}$, that is, $M^* = [\overline{m_{ji}}]$. Denote the maximal ideal space of R equipped with the weak-$*$ topology by $M(R)$. For $x \in R$ denote its Gelfand transform by $\widehat{x}$, that is,

$$\widehat{x}(\varphi) = \varphi(x), \quad \varphi \in M(R), x \in R.$$

For a matrix $M \in R^{p\times m}$ whose entry in the ith row and jth column is denoted by m_{ij}, we define $M^\star \in R^{m\times p}$ to be the matrix whose entry in the ith row and jth column is $m^\star_{ji}$. Also by $\widehat{M}$ we mean the $p \times m$ matrix whose entry in the ith row and jth column is the continuous function $\widehat{m_{ij}}$ on $M(R)$. Summarizing, if $M = [m_{ij}] \in R^{p\times m}$, then

$$M^\star = \left[m^\star_{ji}\right] \in R^{m\times p},$$
$$\widehat{M} = \left[\widehat{m_{ij}}\right] \in (C(M(R);\mathbb{C}))^{p\times m},$$
$$\left(\widehat{M}(\varphi)\right)^* = \left[\overline{\widehat{m_{ji}}(\varphi)}\right] \in \mathbb{C}^{p\times m}.$$

The algebraic question is:

Given $A \in R^{n\times n}$, $B \in R^{m\times n}$, $A \in R^{n\times p}$, under what conditions will the algebraic Riccati equation

$$A^\star Q + QA - QBB^\star Q + C^\star C = 0. \tag{18}$$

have a solution $Q = Q^\star$ in R?

Note that the Gelfand transform of (18) becomes a matrix Riccati equation for each $z \in M(R)$:

$$\widehat{A^\star}(z)\widehat{Q}(z) + \widehat{Q}(z)\widehat{A}(z) - \widehat{Q}(z)\widehat{B}(z)\widehat{B^\star}(z)\widehat{Q}(z) + \widehat{C^\star}(z)\widehat{C}(z) = 0.$$

The idea is to show that the Riccati equation corresponding to the Gelfand transform of (18) has a solution for each $z \in M(R)$ by appealing to known results for matrix Riccati equations and using an implicit function theorem. This led to the following claim in Byrnes [3, Theorem 2.2, p. 248].

Claim 4.3. *Let $A \in R^{n\times n}$, $B \in R^{n\times m}$ and $C \in R^{p\times n}$ be such that for all $\varphi \in M(R)$, $(\widehat{A}(\varphi), \widehat{B}(\varphi))$ and $(\widehat{A^\star}(\varphi), \widehat{C^\star}(\varphi))$ are controllable. Then there exists $Q \in R^{n\times n}$ such that (18) holds.*

The following shows that this claim is false.

Example 4.4. Consider the even-weighted Wiener algebra $W_\alpha(\mathbb{T})$ for arbitrary even weights $(\alpha_k)_{k\in\mathbb{Z}}$ and the involution $\cdot^\sim$. The solution to the Gelfand transform of (18) with $\check{A}(\phi) = 2+\phi$, $\check{B} = 1 = \check{C} \in W_\alpha(\mathbb{T})$ is given by

$$\hat{Q}(z) = \frac{1}{2}\left(4 + z + 1/z + \sqrt{(4+z+1/z)^2+4}\right).$$

Now $(\hat{A}(z), \hat{B}(z))$ is controllable and $(\hat{A}(z), \hat{C}(z))$ is observable for all $z \in \mathbb{C}$. According to the claim by Byrnes we should have a unique solution to the Riccati equation $Q \in W_\alpha(\mathbb{T})$ for arbitrary even weights. However, $\hat{Q}(z)$ has a singularity at $z = -1 \pm \sqrt{\frac{\sqrt{5}-1}{2}} + i\left(-1 \pm \sqrt{\frac{\sqrt{5}+1}{2}}\right)$ and it is only analytic in an annulus contained in the interior of $\mathbb{A}(\beta)$, where $\beta = 2+\sqrt{5} - \sqrt{2\sqrt{5}-2} - \sqrt{2\sqrt{5}+2}$. But the Gelfand transforms of elements in $W_\alpha(\mathbb{T})$ are analytic in $\mathbb{A}(\rho)$. Hence $\hat{Q}$ cannot be in $W_\alpha(\mathbb{T})$ for arbitrary weights.

The correct version of this idea is Theorem 1.4 in [11].

Theorem 4.5. *Let $A \in R^{n\times n}$, $B \in R^{n\times m}$, and $C \in R^{p\times n}$ satisfy the following assumptions for all $z \in M(R)$:*

(A1) $\widehat{(A^\star)}(z) = (\widehat{A}(z))^*$,
(A2) $\widehat{(BB^\star)}(z) = \widehat{B}(z)(\widehat{B}(z))^*$,
(A3) $\widehat{(C^\star C)}(z) = (\widehat{C}(z))^*\widehat{C}(z)$,
(A4) $(\widehat{A}(z), \widehat{B}(z))$ *is stabilizable,*
(A5) $(\widehat{A}(z), \widehat{C}(z))$ *is detectable.*

Then there exists a $Q \in R^{n\times n}$ such that:

1. *Q satisfies (18),*
2. *$A - BB^\star Q$ is exponentially stable,*
3. *for all $z \in M(R)$, $\widehat{Q}(z)$ is positive semidefinite.*

Note that condition (A1) imposes a necessary restriction on the involution: $\cdot^\sim$ satisfies this, but $\cdot^\dagger$ does not. A sufficient condition for all elements to satisfy (A1) (and hence (A2), (A3)) is that the algebra be symmetric. We recall that a unital Banach algebra R with an involution $\cdot^\star$ is *symmetric* if for every $x \in R$, the spectrum of $xx^\star$ (as an element of R) is contained in $[0, +\infty)$.

For symmetric algebras, Theorem 4.5 can be simplified.

Corollary 4.6. *Let R be a commutative unital complex semisimple symmetric Banach algebra with a symmetric involution $\cdot^\star$. Let $A \in R^{n\times n}$, $B \in R^{n\times m}$, and $C \in R^{p\times n}$ satisfy the following assumptions for all $z \in M(R)$:*

(i) $(\widehat{A}(z), \widehat{B}(z))$ *is stabilizable,*
(ii) $(\widehat{A}(z), \widehat{C}(z))$ *is detectable.*

Then there exists a $Q \in R^{n\times n}$ *such that:*

1. Q *satisfies* (18),
2. $A - BB^\star Q$ *is exponentially stable,*
3. $Q = Q^\star$ *and the spectrum of* Q *(as an element of the Banach algebra* $R^{n\times n}$*) is contained in* $[0, +\infty)$.

Similar results can be shown for more general Riccati equations, including H_∞-type Riccati equations (see Sasane [21]).

To apply these results to our class of systems we need to choose the involution $\cdot^\sim$. Then if $W_\alpha(\mathbb{T})$ with this involution satisfies the Gelfand–Raikov–Shilov condition, $W_\alpha(\mathbb{T})$ is a symmetric Banach algebra; otherwise it is not. This means that we can only obtain results on subexponential decay of the Fourier coefficients. From a mathematical point of view the algebraic results are interesting, but from our applications viewpoint they are disappointing. Recall that Corollary 3.2 already offered simple conditions for analyticity in a small annulus around the unit circle. This is better than subexponential decay.

5. Riccati equations on noncommutative Banach algebras

Sections 3 and 4 give sufficient conditions for solutions of Riccati equations for spatially invariant systems to have a spatially decaying property. Of course the same property is desirable for solutions to Riccati equations for distributed systems having the more general form (2), (3). In Motee and Jadbabaie [20] a class of such distributed systems was studied with the goal of finding conditions under which the LQR Riccati equation

$$A^*Q + QA - QBR^{-1}B^*Q + C^*C = 0$$

for spatially distributed systems on the state space $Z = \ell_2(\mathbb{Z}^n)$ will have solutions with an exponential decaying property. Unfortunately, the results claimed in that paper are false (see Curtain [5]), and the solutions need not have an exponential decaying property. However, we can identify classes of spatially distributed systems for which the LQR Riccati equation solutions have a subexponential decaying property. This can be achieved by examining the algebraic properties of the solutions of Riccati equations with entries in noncommutative algebras. The first result in this direction was in 1985 in Bunce [2], where he assumed that the algebra was a C^*-algebra. Unfortunately this is too restrictive for applications to spatially distributed systems, but an idea from the proof in [2] was utilized in Curtain [6] to prove a more general result for algebras satisfying the following:

Assumptions A. $\mathfrak{A}$ is a unital Banach *-algebra which is an inverse-closed subalgebra of $\mathcal{L}(Z)$, where Z is a Hilbert space.

By an inverse-closed subalgebra we mean that it has the inverse-closed property: if $A \in \mathfrak{A}$ has an inverse in $\mathcal{L}(Z)$, then $A^{-1} \in \mathfrak{A}$.

We remark that the inverse-closedness assumption implies that the spectrum of operators in $\mathfrak{A}$ equals their spectrum in $\mathcal{L}(Z)$. Consequently, when e^{At} is exponentially stable on Z, so is e^{A^*t}, and their spectra lie in $\operatorname{Re} s < -\mu$ for some positive μ. Since this also holds for the spectra of A and A^* (considered as elements of $\mathfrak{A}$), we conclude that e^{At} and e^{A^*t} are exponentially stable on $\mathfrak{A}$.

The results hold for the following more general Riccati equation:

$$A^*Q + QA - (QB + N^*)R^{-1}(B^*Q + N) + L = 0, \tag{19}$$

where $A, B, C \in \mathfrak{A}$. We also need the following dual Riccati equation:

$$(A - BR^{-1}N)P + P(A - BR^{-1}N)^* - P(L - NR^{-1}N^*)P + BR^{-1}B^* = 0. \tag{20}$$

We summarize the main results from [6].

Theorem 5.1. *Suppose that:*

- $\mathfrak{A}$ *satisfies Assumption A,*
- $A, B, C \in \mathfrak{A}$,
- *the Riccati equation (19) has a self-adjoint solution* $Q \in \mathcal{L}(Z)$ *such that* $A_Q := A - BR^{-1}N - BR^{-1}B^*Q$ *generates an exponentially stable semigroup,*
- *the dual Riccati equation (20) has a self-adjoint solution* $P \in \mathcal{L}(Z)$,
- $I + QP$ *is invertible.*

Then $Q, P \in \mathfrak{A}$, *and* A_Q *generates an exponentially stable semigroup on* $\mathfrak{A}$ *if at least one of the following additional conditions holds:*

1. $\mathfrak{A}$ *is dense in* $\mathcal{L}(Z)$;
2. $\mathfrak{A} = (\mathfrak{A}_0)^{n\times n}$ *and* $Z = (\mathcal{L}(Z_0))^{n\times n}$, *where* $\mathfrak{A}_0$ *is a* C^**-subalgebra of* $\mathcal{L}(Z_0)$;
3. $\mathfrak{A} = (\mathfrak{A}_0)^{n\times n}$ *and* $Z = (\mathcal{L}(Z_0))^{n\times n}$, *where* Z_0 *is a Hilbert space and* $\mathfrak{A}_0$ *is a unital symmetric Banach* $*$*-subalgebra of* $\mathcal{L}(Z_0)$ *which is inverse-closed and continuously embedded with respect to* $\mathcal{L}(Z_0)$;
4. *there exists* $F \in \mathfrak{A}$ *such that* $A - BR^{-1}N - BR^{-1}B^*F$ *generates an exponentially stable semigroup on* $\mathcal{L}(Z)$.

Note that the extra conditions 1–3 depend only on properties of the algebra, but condition 4 is a stabilizability property with respect to $\mathfrak{A}$. In general, the latter is difficult to prove, so we concentrate on conditions 1–3. Under condition 2 the above theorem generalizes the results in [2] for C^*-algebras. Condition 3 implies that $\mathfrak{A}$ satisfies the inverse-closed Assumption A. In particular, the even-weighted Wiener algebras satisfying the Gelfand–Shilov–Raikov condition satisfy condition 3.

The proof uses an idea from [2] together with connections between the Hamiltonian and the Popov function, and duality properties of certain Riccati equations. It is applicable to the control Riccati equation for the linear quadratic regulator problem ($N = 0$, $L = C^*C$):

$$A^*Q + QA - QBR^{-1}B^*Q + C^*C = 0 \tag{21}$$

where $R > 0$ and is invertible. Its dual equation is called the filter Riccati equation

$$AP + PA^* - PCC^*P + BR^{-1}B^* = 0. \tag{22}$$

Corollary 5.2. *If* $A, B, C \in \mathfrak{A}$, (A, B) *is exponentially stabilizable and* (A, C) *is exponentially detectable with respect to* Z, *then the nonnegative solutions to the Riccati equations (21) and (22) are also in* $\mathfrak{A}$ *provided that at least one of the extra conditions* 1–4 *in Theorem* 5.1 *holds. Moreover,* $A - BR^{-1}B^*Q$ *and* $A - PCC^*$ *generate exponentially stable semigroups on* $\mathfrak{A}$ *and the Popov function* $I + G(-\overline{s})^*G(s)$ *has a spectral factorization over* $\mathfrak{A}$.

In [6, Examples 3.9, 3.11], results similar to Corollary 5.2 are shown for the *positive-real Riccati equations*

$$QA + A^*Q + QBB^*Q + C^*R^{-1}C = 0,$$
$$PA^* + AP + RCR^{-1}CP + BR^{-1}B = 0,$$

and the *bounded-real Riccati equations* with invertible $R = \gamma^2 I - D^*D > 0$ and $S = \gamma^2 I - DD^* > 0$:

$$Q(A + BR^{-1}D^*C) + (A + BR^{-1}D^*C)^*Q + QBB^*Q + C^*(I + DR^{-1}D^*)C = 0,$$
$$P(A + BR^{-1}D^*C)^* + (A + BR^{-1}D^*C)P + PC^*R^{-1}P + B(I + D^*S^{-1}D)B = 0.$$

Note that Corollary 5.2 when applied to the even-weighted Wiener algebras satisfying the Gelfand–Shilov–Raikov condition yields the same result as Corollary 4.6. However, the approach used for noncommutative algebras in [6] is quite different to the approach used for commutative algebras in [11]. In particular, it cannot be used to obtain results for the H_∞-type Riccati equations obtained in [21].

To apply Theorem 5.1 and Corollary 5.2 to spatially distributed systems of the form (2), (3), the first step is to identify a class of subalgebras of $Z = \mathcal{L}(\ell_2(\mathbb{Z}^n))$ which have the inverse-closed property. In Gröchenig and Leinert [15] it was shown that a large class of noncommutative matrix subalgebras of $\mathcal{L}(\ell_2(\mathbb{Z}^d))$ for positive integers n are symmetric algebras with the inverse-closed property.

They are defined by

$$\mathfrak{A}_v^1 := \left\{ \begin{array}{c} A \in \mathcal{L}(\ell_2(\mathbb{Z}^d)) : \sup_{k\in\mathbb{Z}^d} \sum_{l\in\mathbb{Z}^d} |a_{kl}| v(k-l) < \infty, \\ \sup_{l\in\mathbb{Z}^d} \sum_{k\in\mathbb{Z}^d} |a_{kl}| v(k-l) < \infty \end{array} \right\}$$

with norm

$$\|A\|_v := \max \left\{ \sup_{k\in\mathbb{Z}^d} \sum_{l\in\mathbb{Z}^d} |a_{kl}| v(k-l), \sup_{l\in\mathbb{Z}^d} \sum_{k\in\mathbb{Z}^d} |a_{kl}| v(k-l) \right\},$$

where $v : \mathbb{R}^d \to \mathbb{R}^+$ is an *admissible weight function* with the properties:

- $v(x) = e^{\rho(\|x\|)}$ where $\rho : [0, \infty) \to [0, \infty)$ is a continuous concave function with $\rho(0) = 0$;
- $\lim_{k\to\infty} v(kx)^{1/k} = 1$ for all $x \in \mathbb{R}^d$;
- there exist positive constants C, δ, $0 < \delta \le 1$, such that

$$v(x) \ge C(1 + |x|)^\delta.$$

The first condition implies that v is submultiplicative ($v(x+y) \le v(x)v(y)$), while the second condition is a generalized Gelfand–Raikov–Shilov condition. Typical admissible weights are polynomial weights $v(x) = (1 + |x|)^s$, $s > 0$, and subexponential weights $v(x) = e^{\alpha|x|^\beta}$, $\alpha > 0, 0 < \beta < 1$. Note that exponential weights $v(x) = e^{\alpha|x|}$ with $\alpha > 0$ are not admissible. In [15, Theorem 3.1, Corollary 3.2] it is shown that $\mathfrak{A}_v^1$ is a unital symmetric Banach *-algebra which satisfies the inverse-closed assumption in $\mathcal{L}(\ell_2(\mathbb{Z}^d))$. Moreover, $r_{\mathfrak{A}_v^1}(A) = \|A\|_{\mathcal{L}(\ell_2(\mathbb{Z}^d))}$ which implies that $\|A\|_v \ge \|A\|_{\mathcal{L}(\ell_2(\mathbb{Z}^d))}$. Consequently $\mathfrak{A}_v^1$ is continuously embedded in $\mathcal{L}(\ell_2(\mathbb{Z}^d))$. Hence condition 3 of Theorem 5.1 is satisfied, and Theorem 5.1 and Corollary 5.2

are applicable to the Banach *-algebra $(\mathfrak{A}_v^1)^{n\times n}$. Thus Corollary 5.2 gives sufficient conditions for classes of spatially distributed systems on the state space $Z = (\ell_2(\mathbb{Z}^d))^{n\times n}$ to have solutions to the LQR Riccati equation with a spatial decaying property.

Similar results hold for the positive-real and bounded-real Riccati equations (see [6, Examples 3.9, 3.11]).

6. Extensions and open problems

An important class of spatially invariant systems highlighted in [1] is the linear partial differential equations on an infinite interval, for example,

$$\frac{\partial z}{\partial t}(x,t) = \frac{\partial^2 z}{\partial x^2}(x,t) + u(x,t), \quad -\infty < x < \infty, \tag{23}$$

$$z(x,0) = z_0(x). \tag{24}$$

This is a system on the state space $\boldsymbol{L}_2(\imath\mathbb{R})$. Taking formal Fourier transforms yields

$$\hat{z}(\imath\omega,t) = -\omega^2\hat{z}(\imath\omega,t) + \hat{u}(\imath\omega,t) = 0, \qquad \hat{z}(\imath\omega,0) = \hat{z_0}(\imath\omega), \quad \omega\in\mathbb{R}.$$

This can be seen as a system on the state space $\boldsymbol{L}_2(\imath\mathbb{R})$:

$$\hat{z}(t) = A\hat{z}(t) + B\hat{u}(t), \qquad \hat{z}(0) = \hat{z_0},$$

where the generator A is the unbounded multiplicative operator

$$(Af)(\imath\omega) = -\omega^2 f(\imath\omega), \quad f\in \boldsymbol{D}(A) = \{f\in \boldsymbol{L}_2(\imath\mathbb{R}) \mid \omega^2 f(\imath\omega)\in \boldsymbol{L}_2(\imath\mathbb{R})\}.$$

It is a spatially invariant system with an unbounded generator and it is easier to analyze than (23). In fact, it is a standard way of analyzing linear partial differential equations on infinite domains (see Engel and Nagel [13, Chapter VI.5]). This generates a large class of spatially invariant systems on the state space $\boldsymbol{L}_2(\imath\mathbb{R};\mathbb{C}^n)$. Systems on this state space with all operators in $\boldsymbol{L}_\infty(\imath\mathbb{R};\mathbb{C}^{n\times n})$ are already covered by the theory in [11, Example 4.2]. The even-weighted Wiener algebras on the line are subalgebras of $\boldsymbol{L}_\infty(\imath\mathbb{R};\mathbb{C}^{n\times n})$. They have a spatially decaying property and under a Gelfand–Raikov–Shilov-type condition, Riccati equations with entries in the even-weighted Wiener algebra have solutions in the same algebra. However, this theory does not cover spatially invariant systems arising from partial differential equations because they have unbounded generators. This gap is filled in [6, Section 4] where an analogue of Theorem 5.1 under the stabilizability condition 4 with respect to $\mathfrak{A}^{n\times n}$ is proved for systems with unbounded generators of a semigroup on $\mathfrak{A}^{n\times n}$. The open question is how one would check this stabilizability condition. A better approach is to analyze the analyticity properties of the equation in an infinite strip around the imaginary axis using the results from [1,10]. They are analogous to those outlined in Section 3 for a spatially invariant system on the unit circle. While these yield the spatially decaying properties of the Riccati solutions, they do not give information about the stability of the closed-loop operator. This is because pointwise tests for stabilizability (and detectability) are not sufficient for spatially invariant systems on the imaginary axis (see [7,8]). A nice generalization of the tests in Section 2 for spatially invariant systems with unbounded generators is another open problem. Finally, is it possible to prove results on algebraic properties of solutions to H_∞ Riccati equations for noncommutative algebras?

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